

10. Creation and Annihilation Operators

Use tensor products and Fock spaces for describing mathematically the states of many-particle systems.

Folklore

We want to study a mathematical formalism which describes creation and annihilation operators for many-particle systems. We have to distinguish between

- bosons (particles with integer spin like photons, gluons, vector bosons, and gravitons) and
- fermions (particles with half-integer spin like electrons, neutrinos, and quarks).

The point is that the possible number of identical bosons being in the same physical state is unlimited. In contrast to this, the behavior of fermions is governed by the Pauli exclusion principle. This principle tells us that:

Two identical fermions cannot be in the same physical state.

Furthermore, we have the following general principle of indistinguishability for both bosons and fermions:

It is impossible to distinguish between n identical particles.

Roughly speaking, in contrast to planets, elementary particles do not possess any individuality. Fock spaces were introduced by Vladimir Fock (1898–1974) in 1932.¹

10.1 The Bosonic Fock Space

The elements of the bosonic Fock space X are infinite tuples of physical fields. To display the main idea, we restrict ourselves to the prototype of complex-valued fields²

$$\psi : \mathbb{R}^4 \rightarrow \mathbb{C}.$$

¹ V. Fock, Configuration space and second quantization, Z. Phys. **75** (1932), 622–647 (in German). See also P. Jordan and E. Wigner, On the Pauli equivalence principle, Z. Phys. **47** (1928), 631–658 (in German).

² Important generalizations will be studied later on in connection with quantum electrodynamics. This refers to fields

$$\psi : \mathbb{R}^4 \rightarrow \mathbb{C}^m$$

with m components. For describing photons, it will be necessary to pass from Hilbert spaces to indefinite inner product spaces.

The point $x = (x^0, x^1, x^2, x^3)$ describes space and time in an inertial system, that is, the position coordinates x^1, x^2, x^3 are right-handed Cartesian coordinates. Moreover, we introduce the time-like coordinate

$$x^0 := ct$$

where t is time, and c is the velocity of light in a vacuum. The position vector $\mathbf{x} = x^1 \mathbf{i} + x^2 \mathbf{j} + x^3 \mathbf{k}$ refers to the right-handed orthonormal system $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

The Hilbert space $L_2(\mathbb{R}^{4n})$. Choose $n = 1, 2, \dots$. Let us introduce the inner product

$$\langle \psi | \varphi \rangle_n := \int_{\mathbb{R}^{4n}} \psi(x_1, \dots, x_n)^\dagger \varphi(x_1, \dots, x_n) d^4 x_1 \cdots d^4 x_n$$

along with the corresponding norm $\|\psi\|_n := \sqrt{\langle \psi | \psi \rangle_n}$. Hence

$$\|\psi\|_n^2 = \int_{\mathbb{R}^{4n}} |\psi(x_1, \dots, x_n)|^2 d^4 x_1 \cdots d^4 x_n.$$

Here, the arguments $x_1, \dots, x_n$ live in $\mathbb{R}^4$. By definition, the space $L_2(\mathbb{R}^{4n})$ consists of all the functions³

$$\psi : \mathbb{R}^{4n} \rightarrow \mathbb{C}$$

with $\|\psi\|_n < \infty$. The space $L_2(\mathbb{R}^{4n})$ becomes a complex Hilbert space equipped with the inner product $\langle \psi | \varphi \rangle_n$. We have the direct sum decomposition

$$L_2(\mathbb{R}^{4n}) = L_{2,\text{sym}}(\mathbb{R}^{4n}) \oplus L_{2,\text{antisym}}(\mathbb{R}^{4n})$$

where the space $L_{2,\text{sym}}(\mathbb{R}^{4n})$ (resp. $L_{2,\text{antisym}}(\mathbb{R}^{4n})$) contains all the functions

$$\psi = \psi(x_1, \dots, x_n), \quad x_1, \dots, x_n \in \mathbb{R}^4$$

from $L_2(\mathbb{R}^{4n})$ which are symmetric (resp. antisymmetric) with respect to the arguments $x_1, \dots, x_n$.

Tensor products of fields. Suppose that the two functions $\psi, \varphi : \mathbb{R}^4 \rightarrow \mathbb{C}$ live in the Hilbert space $L_2(\mathbb{R}^4)$. Set

$$(\psi \otimes \varphi)(x_1, x_2) := \psi(x_1)\varphi(x_2) \quad \text{for all } x_1, x_2 \in \mathbb{R}^4.$$

Then, the tensor product $\psi \otimes \varphi$ lives in the Hilbert space $L_2(\mathbb{R}^8)$. Introducing the symmetrization

$$\text{sym}(\psi \otimes \varphi) := \frac{1}{2}(\psi \otimes \varphi + \varphi \otimes \psi)$$

and the antisymmetrization

$$\text{antisym}(\psi \otimes \varphi) := \frac{1}{2}(\psi \otimes \varphi - \varphi \otimes \psi) = \frac{1}{2}(\psi \wedge \varphi)$$

of the tensor product $\psi \otimes \varphi$, we get the decomposition

³ We tacitly assume that the functions ψ are measurable with respect to the Lebesgue measure on $\mathbb{R}^{4n}$. This only excludes highly pathological functions having extremely wild discontinuities. In addition, observe that two functions $\psi, \varphi : \mathbb{R}^{4n} \rightarrow \mathbb{C}$ are identified with each other if they only differ on a subset of $\mathbb{R}^{4n}$ which has the $4n$ -dimensional Lebesgue measure zero.

$$\psi \otimes \varphi = \text{sym}(\psi \otimes \varphi) + \text{antisym}(\psi \otimes \varphi)$$

with $\text{sym}(\psi \otimes \varphi) \in L_{2,\text{sym}}(\mathbb{R}^8)$ and $\text{antisym}(\psi \otimes \varphi) \in L_{2,\text{antisym}}(\mathbb{R}^8)$. Moreover, the set of all the finite linear combinations

$$\alpha_1 \psi_1 \otimes \varphi_1 + \dots + \alpha_n \psi_n \otimes \varphi_n, \quad n = 1, 2, \dots$$

with $\psi_1, \varphi_1, \dots \in L_2(\mathbb{R}^4)$ and complex numbers $\alpha_1, \alpha_2, \dots$ is dense in the Hilbert space $L_2(\mathbb{R}^8)$. We write

$$L_2(\mathbb{R}^8) = L_2(\mathbb{R}^4) \otimes L_2(\mathbb{R}^4) = L_2(\mathbb{R}^4)^{\otimes 2}.$$

This is the prototype of the tensor product of two Hilbert spaces. We also write

$$L_{2,\text{antisym}}(\mathbb{R}^8) = L_2(\mathbb{R}^4) \wedge L_2(\mathbb{R}^4) = L_2(\mathbb{R}^4)^{\wedge 2}.$$

For the inner product, we have⁴

$$\langle \varphi_1 \otimes \varphi_2 | \psi_1 \otimes \psi_2 \rangle = \langle \varphi_1 | \psi_1 \rangle \langle \varphi_2 | \psi_2 \rangle.$$

Finally, for two given linear operators $A, B : L_2(\mathbb{R}^4) \rightarrow L_2(\mathbb{R}^4)$, we define the tensor product $A \otimes B$ by setting

$$(A \otimes B)(\psi \otimes \varphi) := A\psi \otimes \varphi + \psi \otimes B\varphi$$

for all $\varphi, \psi \in L_2(\mathbb{R}^4)$.

Definition of the bosonic Fock space. The bosonic Fock space X is defined to be the direct sum

$$X = \bigoplus_{n=0}^{\infty} X_n$$

of the complex Hilbert spaces $X_0 := \mathbb{C}$, $X_1 := L_2(\mathbb{R}^4)$, and

$$X_n := L_{2,\text{sym}}(\mathbb{R}^{4n}), \quad n = 2, 3, \dots$$

Explicitly, this means the following. The bosonic Fock space X consists of all the infinite sequences

$$(\psi_0, \psi_1, \psi_2, \dots)$$

with $\sum_{n=1}^{\infty} \|\psi_n\|_n^2 < \infty$. The function ψ_n is called an n -particle function. Here,

- ψ_0 is an arbitrary complex number,
- the one-particle functions $\psi_1 : \mathbb{R}^4 \rightarrow \mathbb{C}$ are of the form $\psi_1 = \psi_1(x)$ with $x \in \mathbb{R}^4$, and they live in the complex Hilbert space $L_2(\mathbb{R}^4)$,
- the n -particle functions $\psi_n : \mathbb{R}^{4n} \rightarrow \mathbb{C}$,

$$\psi_n = \psi_n(x_1, x_2, \dots, x_n), \quad n = 2, 3, \dots,$$

are symmetric with respect to the n arguments $x_1, x_2, \dots, x_n \in \mathbb{R}^4$, and they live in the complex Hilbert space $L_{2,\text{sym}}(\mathbb{R}^{4n})$.

⁴ Explicitly, the integral $\int (\varphi_1(x_1) \varphi_2(x_2))^{\dagger} \psi_1(x_1) \psi_2(x_2) d^4 x_1 d^4 x_2$ is equal to

$$\int \varphi_1(x_1)^{\dagger} \psi_1(x_1) d^4 x_1 \int \varphi_2(x_2)^{\dagger} \psi_2(x_2) d^4 x_2.$$

The symmetry of the n -particle functions ψ_n reflects the principle of indistinguishability for n bosons. The bosonic Fock space X is an infinite-dimensional complex Hilbert space equipped with the inner product

$$\langle \psi | \varphi \rangle := \psi_0^\dagger \varphi_0 + \sum_{n=1}^{\infty} \langle \psi_n | \varphi_n \rangle_n.$$

10.1.1 The Particle Number Operator

For $\psi := (\psi_0, \psi_1, \psi_2, \dots)$, set

$$N\psi := (0, \psi_1, 2\psi_2, \dots, n\psi_n, \dots).$$

More precisely, the linear operator $N : D(N) \rightarrow X$ is defined for all states $\psi \in X$ of the bosonic Fock space X with

$$\sum_{n=1}^{\infty} n^2 \|\psi_n\|_n^2 < \infty.$$

For example, choose $\psi_n \in X_n$ for fixed index n with $\|\psi_n\|_n = 1$. Define

$$\Psi_n := (0, \dots, 0, \psi_n, 0, 0, \dots)$$

where ψ_n stands at the n th place. Then, Ψ_n is a normalized state in the bosonic Fock space X with

$$N\Psi_n = n\Psi_n.$$

In terms of physics, the state Ψ_n describes n bosons.

10.1.2 The Ground State

The state $|0\rangle := (1, 0, 0, \dots)$ is a normalized state in the bosonic Fock space X with

$$N|0\rangle = 0.$$

The state $|0\rangle$ is called the normalized vacuum state (or briefly the vacuum), since the number of bosons is equal to zero in this state.

Dense linear subspace X_{fin} of the bosonic Fock space X . Let X_{fin} denote the set of all the states

$$\psi = (\psi_0, \psi_1, \dots)$$

in the bosonic Fock space X for which at most a finite number of the functions $\psi_1, \psi_2, \dots$ does not vanish identically. For example, the state

$$\psi = (\psi_0, \psi_1, \dots, \psi_4, 0, 0, \dots)$$

with $\psi_0 \in \mathbb{C}$ and $\psi_j \in L_2(\mathbb{R}^4)$ for $j = 1, \dots, 4$ lies in the subspace X_{fin} .

Composition of particle functions. We are given the one-particle function $f \in L_2(\mathbb{R}^4)$. For each n -particle function

$$\psi_n \in L_{2,\text{sym}}(\mathbb{R}^{4n}),$$

the symmetrization of the tensor product $f \otimes \psi_n$ yields the $(n+1)$ -particle function⁵

$$\varrho_{n+1} := \sqrt{n+1} \cdot \text{sym}(f \otimes \psi_n). \quad (10.1)$$

Intuitively, this is the composition of the one-particle state f with the n -particle state ψ_n . Explicitly,

$$\varrho_{n+1}(x_1, \dots, x_{n+1}) = \frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi} f(x_1) \psi_n(x_2, \dots, x_{n+1})$$

where we sum over all permutations π of the arguments $x_1, \dots, x_{n+1}$.

Creation operator $a^+(f)$. Fix again the one-particle function $f \in L_2(\mathbb{R}^4)$. We want to construct a linear operator

$$a^+(f) : X_{\text{fin}} \rightarrow X$$

which describes the creation of particles. Explicitly, for each sequence

$$\psi := (\psi_0, \psi_1, \psi_2, \dots)$$

in the linear subspace X_{fin} of the bosonic Fock space X , we define

$$a^+(f)\psi := (0, \varrho_1, \varrho_2, \dots) \quad (10.2)$$

where the functions $\varrho_1, \varrho_2, \dots$ are given by (10.1). In particular,

$$\varrho_1(x_1) := f(x_1)\psi_0 \quad \varrho_2(x_1, x_2) = \frac{f(x_1)\psi_1(x_2) + f(x_2)\psi_1(x_1)}{\sqrt{2}}.$$

Annihilation operator $a^-(f)$. We want to construct a linear operator

$$a^-(f) : X_{\text{fin}} \rightarrow X$$

which is formally adjoint to the creation operator $a^+(f)$, that is,

$$\langle a^-(f)\varphi | \varphi \rangle = \langle \varphi | a^+(f)\psi \rangle \quad \text{for all } \varphi, \psi \in X_{\text{fin}}.$$

In other words, we want to get $a^-(f) = (a^+(f))^\dagger$ on X_{fin} . To this end, for each sequence $\varphi := (\varphi_0, \varphi_1, \varphi_2, \dots)$ in X_{fin} we define

$$a^-(f)\varphi := (\chi_0, \chi_1, \chi_2, \dots) \quad (10.3)$$

along with

$$\chi_n(x_1, \dots, x_n) := \sqrt{n+1} \int_{\mathbb{R}^4} f(x)^\dagger \varphi_{n+1}(x, x_1, \dots, x_n) d^4x$$

for all indices $n = 0, 1, 2, \dots$. In particular, we have

$$\chi_0 := \int_{\mathbb{R}^4} f(x)^\dagger \varphi_1(x) d^4x, \quad \chi_1(x_1) := \sqrt{2} \int_{\mathbb{R}^4} f(x)^\dagger \varphi_2(x, x_1) d^4x.$$

⁵ It is convenient to add the normalization factor $\sqrt{n+1}$ (see Theorem 10.1(iii) below).

For the vacuum state, we get

$$a^-(f)|0\rangle = 0 \quad \text{for all } f \in L_2(\mathbb{R}^4).$$

Fundamental commutation relations. Fix $f, g \in L_2(\mathbb{R}^4)$. Recall that $[A, B]_- := AB - BA$. In particular, $[A, B]_- = 0$ is equivalent to $AB = BA$.

Theorem 10.1 *For all states ψ, φ in the linear subspace X_{fin} of the bosonic Fock space X , the following relations hold:*

- (i) *Creation operators:* $[a^+(f), a^+(g)]_- \psi = 0$.
- (ii) *Annihilation operators:* $[a^-(f), a^-(g)]_- \psi = 0$.
- (iii) *Creation and annihilation operators:*

$$[a^-(f), a^+(g)]_- \psi = \langle f|g \rangle_1 \psi. \quad (10.4)$$

- (iv) *Duality:* $\langle a^-(f)\varphi|\psi\rangle = \langle \varphi|a^+(f)\psi\rangle$.

Proof. To display the main ideas of the proof, we restrict ourselves to some special cases. Then the proof of the general case proceeds similarly by induction.

Ad (i)–(iii). Choose the functions $f, g \in L_2(\mathbb{R}^4)$ with $\|f\|_1 = \|g\|_1 = 1$. Since $a^+(g)|0\rangle = (0, g, 0, \dots)$, we have

$$a^+(f)a^+(g)|0\rangle = \frac{1}{\sqrt{2}} (0, 0, f(x_1)g(x_2) + f(x_2)g(x_1), 0, \dots).$$

Using symmetry, $a^+(f)a^+(g)|0\rangle - a^+(g)a^+(f)|0\rangle = 0$.

By $a^-(f)|0\rangle = a^-(g)|0\rangle = 0$, we get $a^-(f)a^-(g)|0\rangle - a^-(g)a^-(f)|0\rangle = 0$. Finally, it follows from

$$a^-(f)a^+(g)|0\rangle = \left(\int_{\mathbb{R}^4} f(x)^\dagger g(x) d^4x, 0, 0, \dots \right)$$

and $a^+(g)a^-(f)|0\rangle = 0$ that

$$a^-(f)a^+(g)|0\rangle - a^+(g)a^-(f)|0\rangle = \langle f|g \rangle_1 |0\rangle.$$

Ad (iv). Choosing the two special states

$$\psi := (0, \psi_1, 0, \dots), \quad \varphi := (0, 0, \varphi_2, 0, \dots),$$

we obtain

$$a^+(f)\psi = \frac{1}{\sqrt{2}} (0, 0, f(x_1)\psi_1(x_2) + f(x_2)\psi_1(x_1), 0, \dots)$$

and

$$a^-(f)\varphi = (0, \sqrt{2} \int_{\mathbb{R}^4} f^\dagger(x_1)\varphi_2(x_1, x_2) d^4x_1, 0, \dots).$$

Therefore, the inner product $\langle \varphi|a^+(f)\psi\rangle$ is equal to

$$\frac{1}{\sqrt{2}} \int_{\mathbb{R}^8} \varphi_2(x_1, x_2)^\dagger \{f(x_1)\psi_1(x_2) + f(x_2)\psi_1(x_1)\} d^4x_1 d^4x_2.$$

Since the function φ_2 is symmetric,

$$\langle \varphi|a^+(f)\psi\rangle = \sqrt{2} \int_{\mathbb{R}^8} \varphi_2(x_1, x_2)^\dagger f(x_1)\psi_1(x_2) d^4x_1 d^4x_2.$$

Furthermore,

$$\langle a^-(f)\varphi|\psi\rangle = \sqrt{2} \left(\int_{\mathbb{R}^8} f^\dagger(x_1)\varphi_2(x_1, x_2) d^4x_1 \right)^\dagger \psi_1(x_2) d^4x_2.$$

Consequently, $\langle a^-(f)\varphi|\psi\rangle = \langle \varphi|a^+(f)\psi\rangle$. $\square$

Physical interpretation. Choose one-particle functions $f_1, \dots, f_s$ in the space $L_2(\mathbb{R}^4)$ such that $\|f_j\|_1 = 1$ for $j = 1, \dots, s$. Set

$$\boxed{\psi := a^+(f_1)a^+(f_2)\cdots a^+(f_s)|0\rangle.} \quad (10.5)$$

This is a state in the bosonic Fock space X . Observe that

$$a^+(f_j)|0\rangle = (0, f_j, 0, \dots)$$

and $Na^+(f_j)|0\rangle = a^+(f_j)|0\rangle$. We say that

- the function f_j represents a normalized one-particle state of a boson, and
- the operator $a^+(f_j)$ generates the normalized one-particle state $a^+(f_j)|0\rangle$ from the vacuum $|0\rangle$.

Note that the state ψ from (10.5) has the form $(\psi_0, \psi_1, \dots)$ where $\psi_j = 0$ if $j \neq s$. Hence

$$\boxed{N\psi = s\psi.}$$

This tells us that if $\psi \neq 0$, then the state ψ from (10.5) represents s bosons being in one-particle states corresponding to $f_1, \dots, f_s$. Because of Theorem 10.1, the state ψ from (10.5) is invariant under permutations of $f_1, \dots, f_s$. This reflects the principle of indistinguishability for s bosons.

Important special case. Consider a system of functions

$$f_1, f_2, f_3, \dots$$

from $\mathbb{R}^4$ to $\mathbb{C}$ which forms an orthonormal system in the Hilbert space $L_2(\mathbb{R}^4)$, that is, $\langle f_k|f_l\rangle_1 = \delta_{kl}$ for all $k, l = 1, 2, \dots$. Define

$$a_j^+ := a^+(f_j), \quad a_j^- := a^-(f_j), \quad j = 1, 2, \dots$$

For all $\psi \in X_{\text{fin}}$ with $j, k = 1, 2, \dots$, we then have the following commutation relations:

$$\boxed{\begin{aligned} [a_j^+, a_k^+]_- \psi &= [a_j^-, a_k^-]_- \psi = 0, \\ [a_j^-, a_k^+]_- \psi &= \delta_{jk}\psi. \end{aligned}} \quad (10.6)$$

This follows immediately from Theorem 10.1. Moreover, for $j, k = 1, 2, \dots$, the following states are normalized in the bosonic Fock space X :

- (i) $a_j^+|0\rangle$;
- (ii) $a_j^+a_k^+|0\rangle$ if $j \neq k$;
- (iii) $\frac{1}{\sqrt{2}}(a_j^+)^2|0\rangle$.

Proof. Ad (i). Note that $a_j^+|0\rangle = (0, f_j, 0, \dots)$.

Ad (ii). We have $a_j^+ a_k^+ |0\rangle = (0, 0, \varphi_2, 0, \dots)$ with

$$\varphi_2(x_1, x_2) := \frac{f_j(x_1)f_k(x_2) + f_j(x_2)f_k(x_1)}{\sqrt{2}}.$$

Since $j \neq k$, $\int_{\mathbb{R}^4} f_j(x)^\dagger f_k(x) d^4x = 0$. Hence $\langle \varphi_2 | \varphi_2 \rangle_2$ is equal to

$$\int_{\mathbb{R}^8} \varphi_2(x, y)^\dagger \varphi_2(x, y) d^4x d^4y = \int_{\mathbb{R}^4} |f_j(x)|^2 d^4x \int_{\mathbb{R}^4} |f_k(y)|^2 d^4y = 1.$$

Ad (iii). Since $j = k$, we obtain

$$\langle \varphi_2 | \varphi_2 \rangle_2 = 2 \int_{\mathbb{R}^4} |f_j(x)|^2 d^4x \int_{\mathbb{R}^4} |f_j(y)|^2 d^4y = 2.$$

This argument finishes the proof. $\square$

More generally, if $1 \leq j_1 < \dots < j_k$ and $m_1, \dots, m_k = 1, 2, \dots$, then

$$\left[\frac{(a_{j_1}^+)^{m_1}}{\sqrt{m_1!}} \frac{(a_{j_2}^+)^{m_2}}{\sqrt{m_2!}} \dots \frac{(a_{j_k}^+)^{m_k}}{\sqrt{m_k!}} |0\rangle \right]$$

is a normalized state in the bosonic Fock space X . States of this form are basic

- in the scattering theory for elementary particles,
- in the theory of many-particle systems in solid state physics, and in
- quantum optics (laser beams).

The rigorous language of operator-valued distributions in quantum field theory. The space of linear operators

$$A : X_{\text{fin}} \rightarrow X$$

is denoted by $L(X_{\text{fin}}, X)$. Set⁶

$$A^+(f) := a^+(f) \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^4).$$

Then, $A^+ : \mathcal{D}(\mathbb{R}^4) \rightarrow L(X_{\text{fin}}, X)$ is a linear map from the space $\mathcal{D}(\mathbb{R}^4)$ of test functions to the operator space $L(X_{\text{fin}}, X)$. That is,

$$A^+(\alpha f + \beta g) = \alpha A^+(f) + \beta A^+(g)$$

for all $f, g \in \mathcal{D}(\mathbb{R}^4)$ and all complex numbers α and β . We call A^+ a distribution with values in the operator space $L(X_{\text{fin}}, X)$. Similarly, we define

$$A^-(f) := a^-(f) \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^4).$$

The map $A^- : \mathcal{D}(\mathbb{R}^4) \rightarrow L(X_{\text{fin}}, X)$ is antilinear, that is,

$$A^-(\alpha f + \beta g) = \alpha^\dagger A^-(f) + \beta^\dagger A^-(g)$$

⁶ Recall that the space $\mathcal{D}(\mathbb{R}^4)$ consists of all the smooth functions $f : \mathbb{R}^4 \rightarrow \mathbb{C}$ which vanish outside some ball, which depends on f . Such functions are called test functions. The space $\mathcal{D}(\mathbb{R}^4)$ is also denoted by $C_0^\infty(\mathbb{R}^4)$.

for all $f, g \in \mathcal{D}(\mathbb{R}^4)$ and all complex numbers α and β . We call A^- an antidistribution with values in the operator space $L(X_{\text{fin}}, X)$.

The formal language of physicists. Physicists introduce the formal creation operators $a^-(x)$ and the formal annihilation operators $a^+(x)$ along with the formal commutation relations

$$\begin{aligned} [a^+(x), a^+(y)]_- &= [a^-(x), a^-(y)]_- = 0, \\ [a^-(x), a^+(y)]_- &= \delta(x - y)I \end{aligned} \quad (10.7)$$

and the duality relations

$$(a^+(x))^\dagger = a^-(x), \quad (a^-(x))^\dagger = a^+(x).$$

These relations are assumed to be valid for all $x, y \in \mathbb{R}^4$. Intuitively, the operator $a^+(x)$ describes the creation of a boson at the given space-time point $x = (ct, \mathbf{x})$. This corresponds to the creation of a boson at the position $\mathbf{x}$ at time t . Similarly, the operator $a^-(x)$ describes the annihilation of a boson at the position $\mathbf{x}$ at time t . Furthermore, we formally write

$$a^+(f) := \int_{\mathbb{R}^4} f(x) a^+(x) d^4x,$$

and $a^-(f) := \int_{\mathbb{R}^4} f(x)^\dagger a^-(x) dx$ along with

$$a^-(f) a^+(g) := \int_{\mathbb{R}^8} f(x)^\dagger g(y) a^-(x) a^+(y) d^4x d^4y,$$

and so on. Mnemonically, this yields the rigorous approach introduced above. For example,

$$\begin{aligned} a^-(f) a^+(g) - a^+(g) a^-(f) &= \int_{\mathbb{R}^8} f(x)^\dagger g(y) [a^-(x), a^+(y)]_- d^4x d^4y \\ &= \int_{\mathbb{R}^8} f(x) g(y) \delta(x - y) I \cdot d^4x d^4y = \left(\int_{\mathbb{R}^4} f(x)^\dagger g(x) d^4x \right) I. \end{aligned}$$

Furthermore,

$$a^+(f) a^+(g) - a^+(g) a^+(f) = \int_{\mathbb{R}^8} f(x)^\dagger g(y) [a^+(x), a^+(y)]_- d^4x d^4y = 0.$$

Similarly, $a^-(f) a^-(g) - a^-(g) a^-(f) = 0$. Finally, we formally get

$$(a^+(f))^\dagger = \left(\int_{\mathbb{R}^4} f(x) a^+(x) d^4x \right)^\dagger = \int_{\mathbb{R}^4} f(x)^\dagger a^-(x) d^4x = a^-(f).$$

10.2 The Fermionic Fock Space and the Pauli Principle

In contrast to the bosonic Fock space, the components $\psi_2, \psi_3, \dots$ of a state in the fermionic Fock space are not symmetric, but antisymmetric functions. As we will see below, this forces the Pauli exclusion principle. Let us consider the

prototype of a fermionic Fock space based on the one-particle function $\psi : \mathbb{R}^4 \rightarrow \mathbb{C}$. The fermionic Fock space Y is defined to be the direct sum

$$Y = \bigoplus_{n=0}^{\infty} Y_n$$

of the complex Hilbert spaces $Y_0 := \mathbb{C}$, $Y_1 := L_2(\mathbb{R}^4)$, and

$$Y_n := L_{2, \text{antisym}}(\mathbb{R}^{4n}), \quad n = 2, 3, \dots$$

Explicitly, the fermionic Fock space Y consists of all the infinite sequences

$$(\psi_0, \psi_1, \psi_2, \dots)$$

with $\sum_{n=1}^{\infty} \|\psi_n\|_n^2 < \infty$. More precisely,

- ψ_0 is an arbitrary complex number,
- the one-particle functions $\psi_1 : \mathbb{R}^4 \rightarrow \mathbb{C}$ are of the form $\psi_1 = \psi_1(x)$ with $x \in \mathbb{R}^4$, and they live in the complex Hilbert space $L_2(\mathbb{R}^4)$,
- the n -particle functions $\psi_n : \mathbb{R}^{4n} \rightarrow \mathbb{C}$,

$$\psi_n = \psi_n(x_1, x_2, \dots, x_n), \quad n = 2, 3, \dots,$$

are antisymmetric with respect to the n arguments $x_1, x_2, \dots, x_n \in \mathbb{R}^4$, and they live in the complex Hilbert space $L_{2, \text{antisym}}(\mathbb{R}^{4n})$.

The antisymmetry of the functions ψ_n reflects the principle of indistinguishability for fermions. The fermionic Fock space Y is an infinite-dimensional complex Hilbert space equipped with the inner product

$$\langle \psi | \varphi \rangle := \psi_0^\dagger \varphi_0 + \sum_{n=1}^{\infty} \langle \psi_n | \varphi_n \rangle_n.$$

Recall that $\langle \psi | \varphi \rangle_n := \int_{\mathbb{R}^{4n}} \psi(x_1, \dots, x_n)^\dagger \varphi(x_1, \dots, x_n) dx_1 \cdots dx_n$.

Particle number operator N . For $\psi := (\psi_0, \psi_1, \psi_2, \dots)$, set

$$N\psi := (0, \psi_1, 2\psi_2, \dots, n\psi_n, \dots).$$

More precisely, the linear operator $N : D(N) \rightarrow Y$ is defined for all states $\psi \in Y$ of the fermionic Fock space Y with

$$\sum_{n=1}^{\infty} n^2 \|\psi_n\|_n^2 < \infty.$$

For example, let $\psi_n \in Y_n$ for fixed index n with $\|\psi_n\|_n = 1$. Choose

$$\Psi_n := (0, \dots, 0, \psi_n, 0, 0, \dots)$$

where ψ_n stands at the n th place. Then, Ψ_n is a normalized state in the fermionic Fock space Y with

$$N\Psi_n = n\Psi_n.$$

In terms of physics, the state Ψ_n describes n fermions.

Vacuum. The state $|0\rangle := (1, 0, 0, \dots)$ is a normalized state in the fermionic Fock space Y with

$$N|0\rangle = 0.$$

The state $|0\rangle$ is called the normalized vacuum state (or briefly vacuum), since the number of fermions is equal to zero in this state.

Dense linear subspace Y_{fin} of the fermionic Fock space Y . Let Y_{fin} denote the set of all states $\psi = (\psi_0, \psi_1, \dots)$ in the fermionic Fock space Y for which at most a finite number of the functions $\psi_1, \psi_2, \dots$ does not vanish identically.

Composition of particle functions. We are given the one-particle function $f \in L_2(\mathbb{R}^4)$. For each n -particle function

$$\psi_n \in L_{2,\text{antisym}}(\mathbb{R}^{4n}),$$

the antisymmetrization of the tensor product $f \otimes \psi_n$ yields the $(n+1)$ -particle function

$$\varrho_{n+1} := \sqrt{n+1} \cdot \text{antisym}(f \otimes \psi_n). \quad (10.8)$$

Intuitively, this is the composition of the one-particle state f with the n -particle state ψ_n . Explicitly,

$$\varrho_{n+1}(x_1, \dots, x_{n+1}) = \frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi} \text{sgn } \pi \cdot f(x_1) \psi_n(x_2, \dots, x_{n+1})$$

where we sum over all permutations π of the arguments $x_1, \dots, x_{n+1}$, and $\text{sgn } \pi$ denotes the sign of the permutation π .

Creation operator $b^+(f)$. Fix again the one-particle function $f \in L_2(\mathbb{R}^4)$. We want to construct a linear operator

$$b^+(f) : Y_{\text{fin}} \rightarrow Y$$

which describes the creation of particles. Explicitly, for each sequence

$$\psi := (\psi_0, \psi_1, \psi_2, \dots)$$

in the linear subspace Y_{fin} of the fermionic Fock space Y , we define

$$b^+(f)\psi := (0, \rho_1, \rho_2, \dots) \quad (10.9)$$

where the functions $\rho_1, \rho_2, \dots$ are given by (10.8). In particular, we have

$$\rho_1(x_1) := f(x_1)\psi_0, \quad \rho_2(x_1, x_2) = \frac{f(x_1)\psi_1(x_2) - f(x_2)\psi_1(x_1)}{\sqrt{2}}.$$

Annihilation operator $b^-(f)$. We want to construct a linear operator

$$b^-(f) : Y_{\text{fin}} \rightarrow Y$$

which is formally adjoint to the creation operator $b^+(f)$, that is,

$$\langle b^-(f)\varphi | \psi \rangle = \langle \varphi | b^+(f)\psi \rangle \quad \text{for all } \varphi, \psi \in Y_{\text{fin}}.$$

In other words, we want to get $b^-(f) = (b^+(f))^\dagger$ on Y_{fin} . To this end, for each sequence $\varphi := (\varphi_0, \varphi_1, \varphi_2, \dots)$ in Y_{fin} , we define

$$b^-(f)\varphi := (\chi_0, \chi_1, \chi_2, \dots) \quad (10.10)$$

along with

$$\chi_n(x_1, \dots, x_n) := \sqrt{n+1} \int_{\mathbb{R}^4} f(x)^\dagger \varphi_{n+1}(x, x_1, \dots, x_n) d^4x$$

for all indices $n = 0, 1, 2, \dots$. In particular, we have

$$\chi_0 := \int_{\mathbb{R}^4} f(x)^\dagger \varphi_1(x) d^4x, \quad \chi_1(x_1) := \sqrt{2} \int_{\mathbb{R}^4} f(x)^\dagger \varphi_2(x, x_1) d^4x.$$

For the vacuum state, we get

$$b^-(f)|0\rangle = 0 \quad \text{for all } f \in L_2(\mathbb{R}^4).$$

Fundamental anticommutation relations. Fix $f, g \in L_2(\mathbb{R}^4)$. Recall that $[A, B]_+ := AB + BA$. In particular, $[A, B]_+ = 0$ is equivalent to the anticommutativity relation $AB = -BA$.

Theorem 10.2 *For all states ψ, φ in the linear subspace Y_{fin} of the fermionic Fock space Y , the following relations hold:*

- (i) *Creation operators:* $[b^+(f), b^+(g)]_+ \psi = 0$.
- (ii) *Annihilation operators:* $[b^-(f), b^-(g)]_+ \psi = 0$.
- (iii) *Creation and annihilation operators:*

$$[b^-(f), b^+(g)]_+ \psi = \langle f|g \rangle_1 \psi. \quad (10.11)$$

- (iv) *Duality:* $\langle b^-(f)\varphi|\psi\rangle = \langle \varphi|b^+(f)\psi\rangle$.

Proof. Let us start with a special case. Choose functions $f, g \in L_2(\mathbb{R}^4)$ with $\|f\|_1 = \|g\|_1 = 1$. Since $b^+(g)|0\rangle = (0, g, 0, \dots)$, we get

$$b^+(f)b^+(g)|0\rangle = \frac{1}{\sqrt{2}} (0, 0, f(x_1)g(x_2) - f(x_2)g(x_1), 0, \dots).$$

By antisymmetry, $b^+(f)b^+(g)|0\rangle + b^+(g)b^+(f)|0\rangle = 0$.

From $b^-(f)|0\rangle = b^-(g)|0\rangle = 0$ we get $b^-(f)b^-(g)|0\rangle + b^-(g)b^-(f)|0\rangle = 0$. Finally, it follows from

$$b^-(f)b^+(g)|0\rangle = \left(\int_{\mathbb{R}^4} f(x)^\dagger g(x) d^4x, 0, 0, \dots \right)$$

and $b^+(g)b^-(f)|0\rangle = 0$ that

$$b^-(f)b^+(g)|0\rangle + b^+(g)b^-(f)|0\rangle = \langle f|g \rangle_1 |0\rangle.$$

The proof of the general case proceeds similarly by induction. $\square$

Physical interpretation. Choose functions $f_1, \dots, f_s \in L_2(\mathbb{R}^4)$ with the normalization condition $\|f_j\|_1 = 1$ for $j = 1, \dots, s$. Set

$$\psi := b^+(f_1)b^+(f_2)\cdots b^+(f_s)|0\rangle. \quad (10.12)$$

This is a state in the fermionic Fock space Y . Observe that

$$b^+(f_j)|0\rangle = (0, f_j, 0, \dots)$$

and $Nb^+(f_j)|0\rangle = b^+(f_j)|0\rangle$. We say that

- the function f_j represents a normalized one-particle state of one fermion, and
- the operator $b^+(f_j)$ generates the normalized one-particle state $b^+(f_j)|0\rangle$ from the vacuum state $|0\rangle$.

In the general case, we get

$$N\psi = s\psi.$$

Therefore, if $\psi \neq 0$, then the state ψ from (10.12) represents s fermions which are in one-particle states corresponding to $f_1, \dots, f_s$.

The Pauli exclusion principle. Because of Theorem 10.2 above, the state ψ from (10.12) changes sign under odd permutations of $f_1, \dots, f_s$. Thus, we get

$$b^+(f_1)b^+(f_2) \cdots b^+(f_s)|0\rangle = 0$$

if two one-particle states f_j and f_k coincide. For example,

$$b^+(f)b^+(f)|0\rangle = 0.$$

Important special case. Consider a system of functions $f_1, f_2, f_3, \dots$ which form an orthonormal system in the Hilbert space $L_2(\mathbb{R}^4)$, that is,

$$\langle f_k | f_l \rangle_1 = \delta_{kl}, \quad k, l = 1, 2, \dots$$

Define

$$b_j^+ := b^+(f_j), \quad b_j^- := b^-(f_j), \quad j = 1, 2, \dots$$

For all $\psi \in Y_{\text{fin}}$ and all $j, k = 1, 2, \dots$, we then have the following anticommutation relations:

$$\boxed{\begin{aligned} [b_j^+, b_k^+]_+ \psi &= [b_j^-, b_k^-]_+ \psi = 0, \\ [b_j^-, b_k^+]_+ \psi &= \delta_{jk} \psi. \end{aligned}} \quad (10.13)$$

If $1 \leq j_1 < \dots < j_k$, then the symbol

$$b_{j_1}^+ b_{j_2}^+ \cdots b_{j_k}^+ |0\rangle$$

represents a normalized state in the fermionic Fock space Y .

The rigorous language of operator-valued distributions. The space of linear operators $B : Y_{\text{fin}} \rightarrow Y$ is denoted by $L(Y_{\text{fin}}, Y)$. Set

$$\boxed{B^+(f) := b^+(f) \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^4).}$$

Then, $B^+ : \mathcal{D}(\mathbb{R}^4) \rightarrow L(X_{\text{fin}}, X)$ is a linear map from the space $\mathcal{D}(\mathbb{R}^4)$ of test functions to the operator space $L(Y_{\text{fin}}, Y)$. We call B^+ a distribution with values in the operator space $L(Y_{\text{fin}}, Y)$. Similarly, we define

$$B^-(f) := b^-(f) \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^4).$$

The map $B^- : \mathcal{D}(\mathbb{R}^4) \rightarrow L(Y_{\text{fin}}, Y)$ is antilinear. We call B^- an antidistribution with values in the operator space $L(Y_{\text{fin}}, Y)$.

The formal language of physicists. Physicists introduce the formal fermionic creation operators $b^+(x)$ and the formal fermionic annihilation operators $b^-(x)$ along with the formal commutation relations

$$\begin{aligned} [b^+(x), b^+(y)]_+ &= [b^-(x), b^-(y)]_+ = 0, \\ [b^-(x), b^+(y)]_+ &= \delta(x - y)I \end{aligned} \quad (10.14)$$

and the duality relations

$$(b^+(x))^\dagger = b^-(x), \quad (b^-(x))^\dagger = b^+(x).$$

These relations are assumed to be valid for all $x, y \in \mathbb{R}^4$. Intuitively, the operator $b^+(x)$ describes the creation of a fermion at the space-time point x (resp. the operator $b^-(x)$ describes the annihilation of a fermion at x). Furthermore, we formally write

$$b^-(f) := \int_{\mathbb{R}^4} f(x)^\dagger b^-(x) d^4x, \quad b^+(f) := \int_{\mathbb{R}^4} f(x) b^+(x) d^4x$$

along with

$$b^-(f)b^+(g) := \int_{\mathbb{R}^8} f(x)^\dagger g(y) b^-(x) b^+(y) d^4x d^4y,$$

and so on. Mnemonically, this yields the rigorous approach introduced above. For example,

$$\begin{aligned} b^-(f)b^+(g) + b^+(g)b^-(f) &= \int_{\mathbb{R}^8} f(x)^\dagger g(y) [b^-(x), b^+(y)]_+ d^4x d^4y \\ &= \int_{\mathbb{R}^8} f(x)^\dagger g(y) \delta(x - y) I \cdot d^4x d^4y = \left(\int_{\mathbb{R}^4} f(x)^\dagger g(x) d^4x \right) I. \end{aligned}$$

Furthermore,

$$b^+(f)b^+(g) + b^+(g)b^+(f) = \int_{\mathbb{R}^8} f(x)g(y) [b^+(x), b^+(y)]_+ d^4x d^4y = 0.$$

Similarly, $b^-(f)b^-(g) + b^-(g)b^-(f) = 0$. Finally, we formally get

$$(b^-(f))^\dagger = \left(\int_{\mathbb{R}^4} f(x)^\dagger b^-(x) d^4x \right)^\dagger = \int_{\mathbb{R}^4} f(x) b^+(x) d^4x = b^+(f).$$

10.3 General Construction

In a straightforward manner, we now want to generalize the construction of bosonic and fermionic Fock spaces to one-particle functions $\psi : \mathbb{R}^4 \rightarrow \mathbb{C}^d$ which possess d degrees of freedom:

$$\psi(x) = \begin{pmatrix} \psi^1(x) \\ \vdots \\ \psi^d(x) \end{pmatrix}, \quad x \in \mathbb{R}^4.$$

We briefly write $\psi(x) = (\psi^j(x))$. The desired generalization can be easily obtained by using systematically the language of tensor products.

The one-particle Hilbert space $L_2(\mathbb{R}^4, \mathbb{C}^d)$. To begin with, let us introduce the following inner product:

$$\langle \psi | \varphi \rangle_1 := \int_{\mathbb{R}^4} \sum_{j=1}^d \psi^j(x)^\dagger \varphi^j(x) d^4 x.$$

By definition, the space $L_2(\mathbb{R}^4, \mathbb{C}^d)$ consists of all the functions $\psi : \mathbb{R}^4 \rightarrow \mathbb{C}^d$ with $\langle \psi | \psi \rangle_1 < \infty$.⁷ The space $L_2(\mathbb{R}^4, \mathbb{C}^d)$ becomes a complex Hilbert space equipped with the inner product $\langle \psi | \varphi \rangle_1$.

Bosonic two-particle functions. Let $\psi, \varphi \in L_2(\mathbb{R}^4, \mathbb{C}^d)$ be one-particle functions. The prototype of a two-particle function is the tensor product $\psi \otimes \varphi$. Explicitly, this is the tuple

$$(\psi \otimes \varphi)(x_1, x_2) := (\psi^i(x_1) \varphi^j(x_2))_{i,j=1,\dots,d}, \quad x_1, x_2 \in \mathbb{R}^4.$$

Naturally enough, the inner product is defined by

$$\langle \psi \otimes \varphi | \psi_* \otimes \varphi_* \rangle_2 := \int_{\mathbb{R}^8} \sum_{i,j=1}^d \Psi^{ij}(x_1, x_2)^\dagger \Psi_*^{ij}(x_1, x_2) d^4 x_1 d^4 x_2$$

where $\Psi^{ij}(x_1, x_2) := \psi^i(x_1) \varphi^j(x_2)$, and $\Psi_*^{ij}(x_1, x_2) := \psi_*^i(x_1) \varphi_*^j(x_2)$, and

In order to get a bosonic two-particle function, we have to symmetrize. This means that we have to pass from $\psi \otimes \varphi$ to

$$\text{sym}(\psi \otimes \varphi) := \frac{1}{2}(\psi \otimes \varphi + \varphi \otimes \psi).$$

In general, by a bosonic two-particle function we understand a tuple

$$\Psi(x_1, x_2) = (\Psi^{ij}(x_1, x_2))_{i,j=1,\dots,d}, \quad x_1, x_2 \in \mathbb{R}^4,$$

which is symmetric with respect to both the indices i, j and the arguments x_1, x_2 . Explicitly, we obtain

$$\Psi^{ij}(x_1, x_2) = \Psi^{ji}(x_2, x_1), \quad i, j = 1, \dots, d, \quad x_1, x_2 \in \mathbb{R}^4.$$

In addition, we assume that all the components Ψ^{ij} live in the space $L_2(\mathbb{R}^8)$. We briefly write

$$\Psi \in L_{2,\text{sym}}(\mathbb{R}^8, \mathbb{C}^{d^2}).$$

In particular, for the bosonic two-particle functions $\Psi, \Phi \in L_{2,\text{sym}}(\mathbb{R}^8, \mathbb{C}^{d^2})$, the inner product is given by

$$\langle \Psi | \Phi \rangle_2 := \int_{\mathbb{R}^8} \sum_{i,j=1}^d \Psi^{ij}(x_1, x_2)^\dagger \Phi^{ij}(x_1, x_2) d^4 x_1 d^4 x_2.$$

Fermionic two-particle functions. We now replace symmetry by antisymmetry. For given one-particle functions $\psi, \varphi \in L_2(\mathbb{R}^4, \mathbb{C}^d)$, antisymmetrization yields the special fermionic two-particle function

$$\text{antisym}(\psi \otimes \varphi) = \frac{1}{2}(\psi \otimes \varphi - \varphi \otimes \psi) = \frac{1}{2}(\psi \wedge \varphi).$$

⁷ We tacitly assume that the components of the functions ψ are measurable with respect to the Lebesgue measure on $\mathbb{R}^4$. In addition, observe that two functions $\psi, \varphi : \mathbb{R}^4 \rightarrow \mathbb{C}^d$ are identified with each other if they only differ on a subset of $\mathbb{R}^4$ which has the 4-dimensional Lebesgue measure zero.

Generally, by a fermionic two-particle function we understand a tuple

$$\Psi(x_1, x_2) = (\Psi^{ij}(x_1, x_2)), \quad i, j = 1, \dots, d, \quad x_1, x_2 \in \mathbb{R}^4$$

which is antisymmetric with respect to both the indices i, j and the arguments x_1, x_2 . Explicitly,

$$\Psi^{ij}(x_1, x_2) = -\Psi^{ji}(x_2, x_1), \quad i, j = 1, \dots, d, \quad x_1, x_2 \in \mathbb{R}^4.$$

In addition, we assume that all the components Ψ^{ij} live in the space $L_2(\mathbb{R}^8)$. We briefly write $\Psi \in L_{2, \text{antisym}}(\mathbb{R}^8, \mathbb{C}^{d^2})$. Next we want to introduce

- the bosonic Fock space, and
- the fermionic Fock space.

The bosonic Fock space. The direct sum

$$X = \bigoplus_{n=0}^{\infty} X_n$$

of the Hilbert spaces $X_0 := \mathbb{C}$, $X_1 := L_2(\mathbb{R}^4, \mathbb{C})$, and

$$X_n := L_{2, \text{sym}}(\mathbb{R}^{4n}, \mathbb{C}^{d^n}), \quad n = 2, 3, \dots$$

is called the bosonic Fock space to the one-particle Hilbert space $L_2(\mathbb{R}^4, \mathbb{C}^d)$. Let $i_1, \dots, i_n = 1, \dots, d$ and $x_1, \dots, x_n \in \mathbb{R}^4$. By definition, the elements of the space X_n are tuples

$$\Psi(x_1, \dots, x_n) = (\Psi^{i_1 \dots i_n}(x_1, \dots, x_n))$$

which are symmetric with respect to both the indices $i_1, \dots, i_n = 1, \dots, d$ and the n space-time variables $x_1, \dots, x_n$. Moreover, all of the components $\Psi^{i_1 \dots i_n}$ live in the space $L_2(\mathbb{R}^{4n})$. The elements of the bosonic Fock space X are infinite tuples

$$\Psi = (\Psi_0, \Psi_1, \dots)$$

where Ψ_0 is a complex number, and $\Psi_n \in X_n$ for $n = 1, 2, \dots$. In addition, we postulate that $\sum_{n=1}^{\infty} \langle \Psi_n | \Psi_n \rangle_n < \infty$ where we define

$$\langle \Psi_n | \Phi_n \rangle_n := \int_{\mathbb{R}^{4n}} \sum_{i_1, \dots, i_n=1}^d (\Psi^{i_1 \dots i_n})^\dagger \Phi^{i_1 \dots i_n} d^4 x_1 \dots d^4 x_n.$$

The bosonic Fock space X becomes a complex Hilbert space equipped with the inner product

$$\langle \Psi | \Phi \rangle := \Psi_0^\dagger \Phi_0 + \sum_{n=1}^{\infty} \langle \Psi_n | \Phi_n \rangle_n.$$

The linear subspace X_{fin} and the vacuum state $|0\rangle := (1, 0, 0, \dots)$ are defined as in Sect. 10.1. For each given one-particle function $f \in X_1$, the creation operator

$$a^+(f) : X_{\text{fin}} \rightarrow X$$

is defined by $a^+(f)\Psi := (0, \rho_1, \rho_2, \dots)$ where

$$\rho_{n+1} := \sqrt{n+1} \cdot \text{sym}(f \otimes \Psi_n), \quad n = 0, 1, 2, \dots$$

Explicitly, $\varrho_{n+1}^{i_1 \dots i_{n+1}}(x_1, \dots, x_{n+1})$ is equal to

$$\frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi} \pi(f^{i_1}(x_1)) \Psi_n^{i_2 \dots i_{n+1}}(x_2, \dots, x_{n+1})$$

where we sum over all permutations π of $1, \dots, n+1$. The operation $\pi(\dots)$ refers to permutations of both the indices $i_1, \dots, i_{n+1}$ and the arguments $x_1, \dots, x_{n+1}$. The annihilation operator

$$a^-(f) : X_{\text{fin}} \rightarrow X$$

is the formally adjoint operator to the creation operator $a^+(f)$, that is,

$$\langle a^-(f)\Phi | \Psi \rangle = \langle \Phi | a^+(f)\Psi \rangle \quad \text{for all } \Phi, \Psi \in X_{\text{fin}}.$$

In other words, $a^-(f) = (a^+(f))^{\dagger}$ on X_{fin} . Explicitly, for each given sequence $\Phi := (\Phi_0, \Phi_1, \Phi_2, \dots)$ in X_{fin} , we define

$$a^-(f)\psi := (\chi_0, \chi_1, \chi_2, \dots)$$

where $\chi_n^{i_1 \dots i_n}(x_1, \dots, x_n)$ is given by

$$\sqrt{n+1} \int_{\mathbb{R}^4} \sum_{i=1}^d f^i(x)^{\dagger} \Phi_{n+1}^{i i_1 \dots i_n}(x, x_1, \dots, x_n) d^4 x.$$

In particular, $\chi_0 = \int_{\mathbb{R}^4} \sum_{i=1}^d f^i(x)^{\dagger} \Psi^i(x) d^4 x$.

The fermionic Fock space. The direct sum

$$Y = \bigoplus_{n=0}^{\infty} Y_n$$

of the Hilbert spaces $Y_0 := \mathbb{C}$, $Y_1 := L_2(\mathbb{R}^4, \mathbb{C})$, and

$$Y_n := L_{2, \text{antisym}}(\mathbb{R}^{4n}, \mathbb{C}^{d^n}), \quad n = 2, 3, \dots$$

is called the fermionic Fock space to the one-particle Hilbert space $L_2(\mathbb{R}^4, \mathbb{C}^d)$. Let $i_1, \dots, i_n = 1, \dots, d$ and $x_1, \dots, x_n \in \mathbb{R}^4$. By definition, the elements of the space Y_n are tuples

$$\Psi(x_1, \dots, x_n) = (\Psi^{i_1 \dots i_n}(x_1, \dots, x_n))$$

which are antisymmetric with respect to both the indices $i_1, \dots, i_n$ and the n space-time variables $x_1, \dots, x_n$. Moreover, the components $\Psi^{i_1 \dots i_n}$ live in the space $L_2(\mathbb{R}^{4n})$. Explicitly, the elements of the fermionic Fock space Y are infinite tuples

$$\Psi = (\Psi_0, \Psi_1, \dots)$$

where Ψ_0 is a complex number, and $\Psi_n \in Y_n$ for $n = 1, 2, \dots$. In addition, we postulate that $\sum_{n=1}^{\infty} \langle \Psi_n | \Psi_n \rangle_n < \infty$. The space Y becomes a complex Hilbert space equipped with the inner product

$$\langle \Psi | \Phi \rangle := \Psi_0^{\dagger} \Phi_0 + \sum_{n=1}^{\infty} \langle \Psi_n | \Phi_n \rangle_n.$$

For each given one-particle function $f \in Y_1$, the creation operator

$$b^+(f) : Y_{\text{fin}} \rightarrow Y$$

is defined by $b^+(f)\Psi := (0, \rho_1, \rho_2, \dots)$ where

$$\boxed{\rho_{n+1} := \sqrt{n+1} \cdot \text{antisym}(f \otimes \Psi_n), \quad n = 0, 1, 2, \dots}$$

Explicitly, $\varrho_{n+1}^{i_1, \dots, i_{n+1}}(x_1, \dots, x_{n+1})$ is equal to

$$\frac{\sqrt{n+1}}{(n+1)!} \sum_{\pi} \text{sgn } \pi \cdot \pi(f^{i_1}(x_1) \Psi_n^{i_2 \dots i_{n+1}}(x_2, \dots, x_{n+1}))$$

where we sum over all permutations π of $1, \dots, n+1$. The operation $\pi(\dots)$ refers to permutations of both the indices $i_1, \dots, i_{n+1}$ and the arguments $x_1, \dots, x_{n+1}$. The annihilation operator

$$b^-(f) : Y_{\text{fin}} \rightarrow Y$$

is the formally adjoint operator to the creation operator $b^+(f)$, that is,

$$\boxed{\langle b^-(f)\Phi | \Psi \rangle = \langle \Phi | b^+(f)\Psi \rangle \quad \text{for all } \Phi, \Psi \in Y_{\text{fin}}.}$$

In other words, $b^-(f) = (b^+(f))^{\dagger}$ on Y_{fin} . Explicitly, for each sequence

$$\Phi := (\Phi_0, \Phi_1, \Phi_2, \dots)$$

in the space Y_{fin} , we define

$$b^-(f)\Phi := (\chi_0, \chi_1, \chi_2, \dots)$$

where $\chi_n^{i_1 \dots i_n}(x_1, \dots, x_n)$ is equal to

$$\sqrt{n+1} \int_{\mathbb{R}^4} \sum_{i=1}^d f^i(x) \Phi_{n+1}^{i i_1 \dots i_n}(x, x_1, \dots, x_n) d^4 x.$$

10.4 The Main Strategy of Quantum Electrodynamics

The most important experiments in elementary particle physics are scattering experiments carried out in huge high-energy particle accelerators. Physicists characterize the outcome of such experiments by cross sections. If $J = \varrho v$ is the current density of the incoming particle stream with velocity v and particle density ϱ , then

$$N = \sigma T \cdot J$$

is the number of scattered particles observed during the time interval $[-\frac{T}{2}, \frac{T}{2}]$. In the SI system of physical units, J has the physical dimension of particle density times velocity, $1/\text{m}^2\text{s}$. Therefore, the quantity σ has the physical dimension of area, m^2 , and σ is called the *cross section* of the scattering process. Observe the following:

- Cross sections follow from transition probabilities.
- Transition probabilities result from transition amplitudes.

- Transition amplitudes can be computed by using Feynman diagrams and the corresponding Feynman rules.

Our goal is to motivate the Feynman rules in quantum electrodynamics and to apply them to the computation of cross sections.

The Feynman rules represent the hard core of quantum field theory.

For quantum electrodynamics, the Feynman rules will be summarized in Sect. 14.3. Applications to scattering processes can be found in Chap. 15.

Quantum electrodynamics studies the interaction between the following particles: electrons, positrons, and photons. Here, photons represent quantized electromagnetic waves. Note that:

- The electron is called the basic particle of quantum electrodynamics.
- The positron is the antiparticle to the electron.
- The massless photon is responsible for the interaction between electrons and positrons. Therefore, photons are called the interacting particles of quantum electrodynamics.

In order to understand quantum field theory, one has to start with quantum electrodynamics. Let us discuss the main ideas of quantum electrodynamics. We will proceed in the following four steps:

- (C) Classical field theory: We first consider the classical principle of critical action for the Maxwell–Dirac field which is obtained by coupling the classical electromagnetic field to the Dirac field for the relativistic electron.
- (F) The free quantum field: For the free electromagnetic field and the free Dirac field of the electron, we find solutions in the form of finite Fourier series. Replacing Fourier coefficients by creation and annihilation operators, we get the corresponding free quantum fields for electrons, positrons, and photons (the method of Fourier quantization).

These free quantum fields depend on the choice of both a finite box in position space and a finite lattice in momentum space.

- (I) The interacting quantum field: We use the interaction term between the electrodynamic field and the Dirac field for the electron in order to formulate the Dyson series for the S -matrix of quantum electrodynamics. The S -matrix is a formal power series expansion with respect to the dimensionless coupling constant in the SI system of physical units:

$$\boxed{\kappa := \sqrt{4\pi\alpha}.} \quad (10.15)$$

Here, α denotes the so-called fine structure constant in quantum electrodynamics:

$$\boxed{\alpha = \frac{1}{137.04} = 0.007297} \quad (10.16)$$

which is dimensionless. In addition, $-e$ is the negative electric charge of the electron.⁸ In the SI system, we have

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}.$$

⁸ If we want to emphasize that κ and α refer to quantum electrodynamics, but not to strong and weak interaction in the Standard Model, then we write κ_{QED} and α_{QED} , respectively.

The point is that the Dyson series for the S -matrix depends nonlinearly on the free quantum fields for electrons, positrons, and photons. Using the approximation of the S -matrix in lowest nontrivial order, we are able to compute approximately scattering processes for electrons, positrons, and photons.

- (R) Renormalization: Using higher-order approximations of the S -matrix together with the high-energy limit (resp. the low-energy limit), we get divergent expressions for scattering processes. In order to extract physical information from those divergent expressions, we have to use the crucial method of renormalization. The final results are cross sections for scattering processes of the form

$$\sigma = \sigma_1 \kappa + \sigma_2 \kappa^2 + \dots$$

This is a power series expansion with respect to the small dimensionless coupling constant κ given by (10.15). The coefficients $\sigma_1, \sigma_2, \dots$ are real numbers (equipped with the physical dimension of area) coming from divergent integrals by using a regularization procedure. The coincidence between theory and physical experiment is extremely precise in quantum electrodynamics.

The smallness of the dimensionless (electromagnetic) fine structure constant α is responsible for the incredible success of perturbation theory in quantum electrodynamics.

The situation changes completely in strong interaction where the coupling constant is approximately equal to one, $\kappa = 1$. Then the results of perturbation theory are only crude approximations of reality.

In string theory, there exists a duality transformation between certain models which allows us to transform some models having large coupling constant into dual models having small coupling constant. In the future, physicists hope to establish such a beautiful duality method for strong interaction in nature.

Convention for the choice of the system of physical units. To simplify notation, in the following chapters we will use the energetic system of units, that is, we set

$$\hbar = c = \varepsilon_0 = \mu_0 = k := 1. \quad (10.17)$$

Then, the dimension of an arbitrary physical quantity is some power of energy (see the Appendix A.2 of Vol. I). In particular, the electric charge $-e$ of the electron is dimensionless, and we have

$$e = \sqrt{4\pi\alpha}.$$

The gauge condition. It is a typical feature of quantum electrodynamics that we do not start with the electromagnetic field $\mathbf{E}, \mathbf{B}$, but with the four-potential $U, \mathbf{A}$. The electromagnetic field is then given by

$$\mathbf{E} = -\text{grad } U - \dot{\mathbf{A}}, \quad \mathbf{B} = \text{curl } \mathbf{A}.$$

The point is that the four-potential is only determined up to a gauge transformation of the form

$$U \mapsto U - \frac{\partial f}{\partial t}, \quad \mathbf{A} \mapsto \mathbf{A} + \text{grad } f$$

where f is a smooth function. This causes some trouble. We will overcome the difficulties in the following sections by using the following trick:

- (i) We first destroy the gauge invariance by passing to a modified Lagrangian.
- (ii) The corresponding free quantum fields include virtual photons which do not possess an obvious physical meaning.

- (iii) In classical theory, virtual photons are eliminated by adding the Lorenz gauge condition.
- (iv) In quantum field theory, virtual photons are eliminated by adding a weak Lorenz gauge condition (Gupta–Bleuler quantization).

Nevertheless, we will see that virtual photons essentially influence physical processes proceeding in our real world. The point is that there arise terms in perturbation theory which depend on the photon propagator, and this photon propagator contains contributions coming from virtual photons. In general, quantum electrodynamics adds new physical effects to classical electrodynamics which can be summarized under the sketch word *quantum fluctuations of the ground state* (also called the vacuum). In particular, this concerns the so-called vacuum polarization.